Lecture 27

- $A\vec{x} = \vec{b}$ always has a least squares solution $\vec{x}^*$

  i.e. $A^T A \vec{x}^* = A^T \vec{b}$

  - Least squares solution is unique i.f.f. $A\vec{x} = \vec{0}$ has a unique solution.

    ↳ So they are not ALWAYS unique

Determinants (6.1/6.2)

Ex

- What is the area of the parallelogram P with vertices $\vec{0}$, $\underset{\begin{pmatrix}4\\12\end{pmatrix}}{\vec{v}_1}$, $\underset{\begin{pmatrix}-6\\9\end{pmatrix}}{\vec{v}_2}$, $\vec{v}_1 + \vec{v}_2$

- Also $P = \{ T_1 \vec{v}_1 + T_2 \vec{v}_2 \in \mathbb{R}^2 : 0 \le T_1, T_2 \le 1 \}$

Solution: Let's call the area of P

$$\det\begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} = \det\begin{pmatrix} 4 & 12 \\ -6 & 9 \end{pmatrix} = 4\det\begin{pmatrix} 1 & 3 \\ -6 & 9 \end{pmatrix} = (-3)(4)\det\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix}$$

Row operation

$$= (-12)\det\begin{pmatrix} 1 & 3 \\ 0 & -9 \end{pmatrix} \rightarrow (-12)(-9)\det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = 108\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = 108\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 108(1) = 108$$

☆ - A row operation in determinant does not change the area.

• Let $M_{n\times n} = \{\text{all } n\times n \text{ matrices}\}$

<u>Def</u>: A <u>determinant</u> is a fn $\det : M_{n\times n} \to \mathbb{R}$ with the following properties:

- $\det(I) = 1$
- if B is obtained fro A by multiplying a row of A by a scaler $c \in \mathbb{R}$ then $\det(B) = c\det(A)$
- if B is obtained from A by adding a multiple of one row of A to another, then $\det(B) = \det(A)$

$\boxed{\text{Ex}}$ Find det of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 13 \end{bmatrix}$

Solution $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 13 \end{vmatrix} \begin{matrix} \\ -4R_1 \\ -7R_1 \end{matrix}$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -8 \end{vmatrix} \cdot \left(-\tfrac{1}{3}\right)$$

$$= (-3)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{vmatrix}$$

$$= (-3)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2(-3)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

When you get to a diagonal of all ones and zeros underneath the determinent is always one.

$$= 2(-3)\cdot(1)$$

$$\boxed{= -6}$$

Obs: From the definition of det f_n, I can compute it. I don't yet know $\exists$ such a f_n.

• What other properties must a det f_n have?

Prop: Given an $n \times n$ matrix, we have:

(1) $\det A = 0$, if some row of A is 0

(2) more generally $\det A = 0$ if rows of A are linearly dependent

(3) If B is obtained from A by swapping 2 rows of A, then $\det(B) = -\det(A)$